Functional Analysis and Optimization Problems in Hydrodynamic Propulsion

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1. Introduction

Hydrodynamic propulsion is of interest in the biological sciences for the study of swimming creatures, but it is also important in technics, already since ships came into use. We direct here our attention to the propulsion of ships as it is studied at (technical) universities, at ship research institutes and sometimes at shipyards and screw factories. The research described below was inspired by the desire in the shipbuilding industry in the late 1970's to diminish propulsion costs in view of rising energy prices. In general the theoretical research is directed to the solution of practical problems. Analysis in the form of 'classical' applied mathematics in combination with extensive computer programs is employed for the application of lifting surface theories to propellers. By a propeller we mean not only the well-known screw propeller, but also periodically moving wings which cause a thrust. An example of the latter is the Voith-Schneider propeller.

Most propellers are placed at the stern of a ship for the following reason. By its slight viscosity the water flowing along the hull of the ship is dragged with it and obtains kinetic energy with respect to the water at large distance. It can be shown that by placing the propeller at the stern of the ship, part of this kinetic energy can be regained by which the efficiency of the propeller increases.

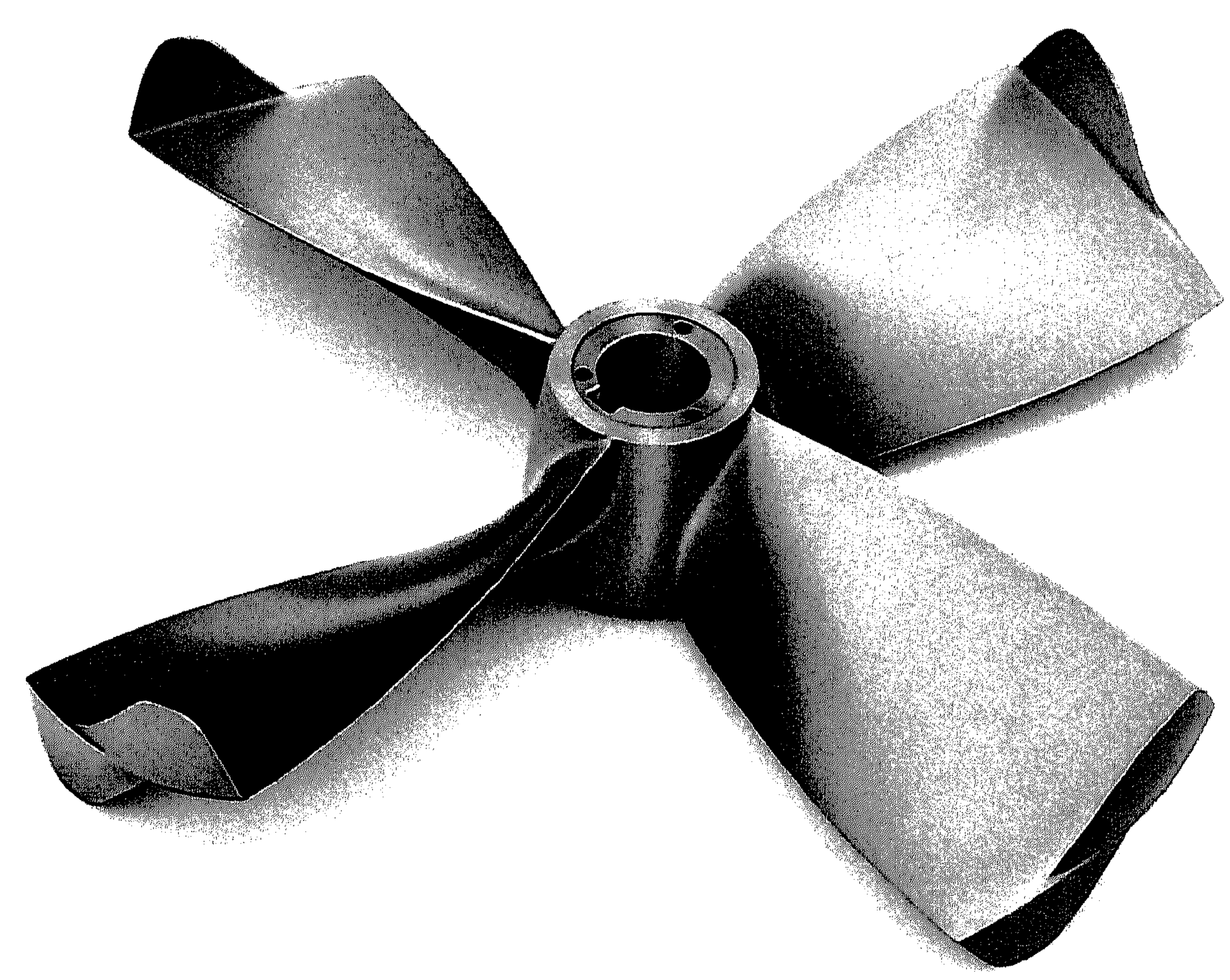

Figure 1. Four-bladed screw propeller with endplates. (Photo: Groningen Propeller Technology B.V., The Netherlands.)

We now mention two difficulties for the calculation of the performance of the propeller which are caused by its above mentioned efficiency increasing position at the stern. First, at the stern of a ship the flow of the water is 'untidy'. The water dragged with the hull becomes turbulent and because it has to follow the shape of the hull, it has to converge at the stern. Besides this the wave pattern at the free surface above the propeller causes a velocity field which varies with depth. Second, the stern forms partly a rigid boundary of the flow domain and hampers the water to be set into motion by the propeller. The same holds for the rudder and also the free surface acts as part of the boundary of the region in which the propeller operates.

Besides the foregoing ones, another type of difficulty can occur, perhaps more specifically with respect to the screw propeller, namely when the propeller is heavily loaded. Then the interaction between vortices (which causes the nonlinear roll-up of the shed vorticity) becomes important and also time-dependent cavitation can be present at the blades.

These are not ideal circumstances for the application of elegant mathematics in order to describe the performance of the propeller. For that sake we have to make simplifications. However, in that case we have to be careful with the application of the results to reality. A rather accessible situation occurs when we suppose that the propeller acts in an incompressible, inviscid, unbounded and otherwise undisturbed fluid and translates with constant velocity and delivers a prescribed thrust. Often it is also assumed that this thrust is sufficiently small so that a (semi-)linearized theory can be used in which squares of velocities induced by the shed vorticity can be neglected with respect to these velocities themselves.

Hence, one of the simplifications is the neglect of viscosity of the fluid. However, loss of efficiency of the propeller caused by viscosity can be very important, especially with respect to optimization problems. Luckily we can introduce in an inviscid optimization theory experimentally or theoretically obtained results of the viscous resistance of plates, by which the viscosity can often be incorporated satisfactorily.

In technically useful optimization calculations it is, for instance with respect to the screw propeller, not always necessary to use functional analytic methods, because by experience built up in the course of time it is known that optimum screw propellers do exist under certain simple constraints. However, there are propeller types for which we are not sure that an optimum propeller does exist within a 'set' of admitted propellers. This can happen rather easily with propellers consisting of periodically moving thrust producing wings, of which we shall discuss two examples. Both examples are two-dimensional, because besides the mentioned simplifications it is also assumed that the wings are infinitely long. First, the small amplitude motion of a thrust producing flat profile and, second, the large amplitude motion of a lifting line (i.e., a line on which the forces are assumed to be concentrated) for which there is an unequality constraint on its lateral force action. It is clear that both models are highly idealized versions of a practically possible propeller. Nevertheless it is very elucidating to understand their working in the simplified case.

2. Optimization of small amplitude motions of a flat profile

We shall discuss small amplitude motions of a rigid profile through a previously undisturbed fluid [1]. With respect to the Cartesian coordinate system (x, y) shown in figure 2 the motion is given by

$$y = h(x,t) = \frac{\ell}{2}a(t) + \alpha(t)(x - Ut), \quad -\ell \leq x \leq \ell, \tag{2.1}$$

where 2ℓ is the length of the profile, $\frac{\ell}{2}a(t)$ and $\alpha(t)$ are the so-called heaving and pitching parts of the motion, and U is the constant velocity of the profile in the positive x-direction.

Let $T(h)$ be the mean thrust generated by the periodic motion h with period τ_0 and let $E(h)$ be the mean increase of kinetic energy of the fluid

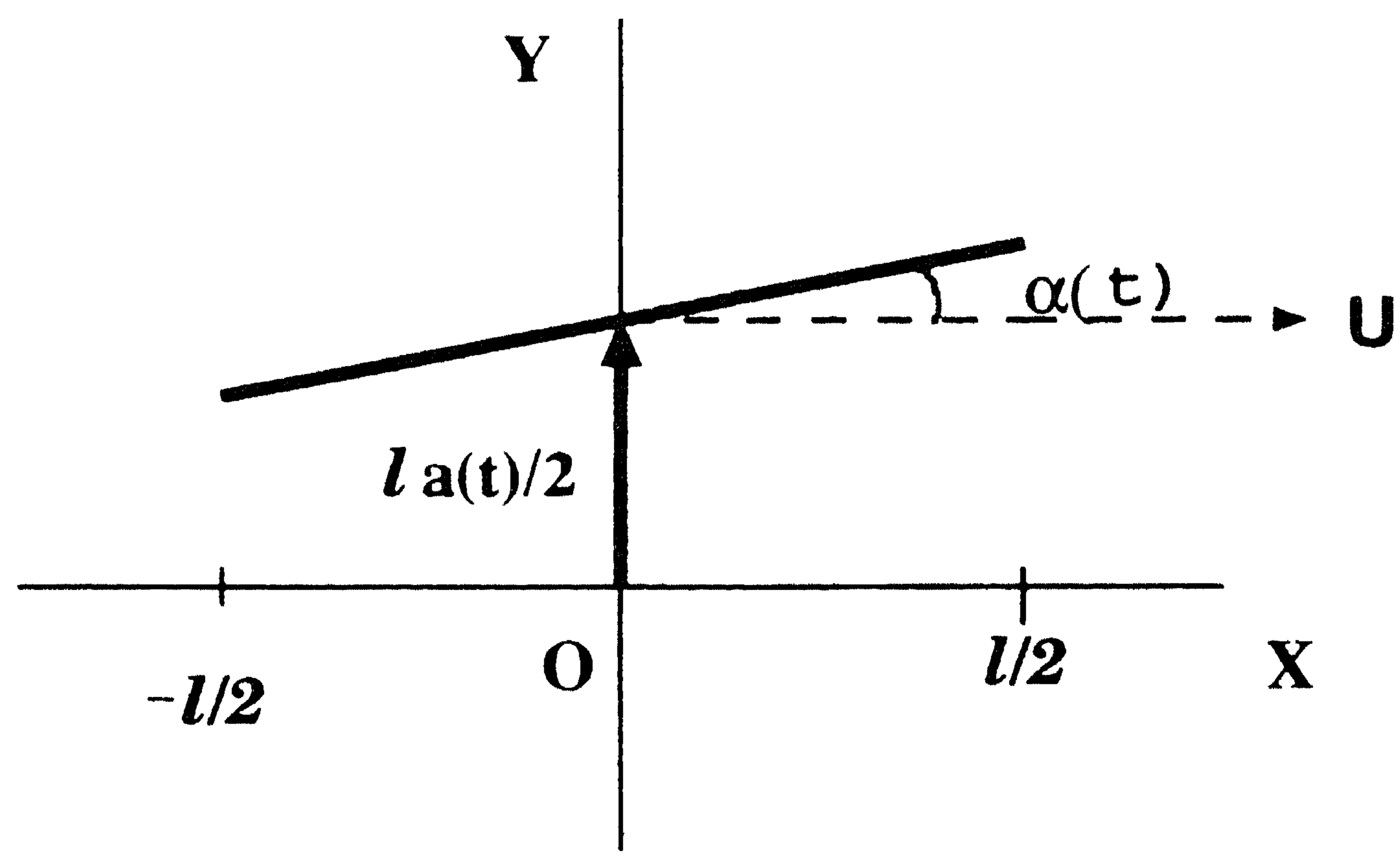

Figure 2. Flat profile of length 2ℓ moving through a previously undisturbed fluid at velocity U.

during one period. The efficiency of the motion h is useful work divided by total work, hence

$$\eta(h) = \frac{UT(h)}{UT(h) + E(h)} \cdot \tag{2.2}$$

The aim of the optimization is to minimize the lost energy E subject to the constraint that a prescribed mean thrust $\overline{T}$ is generated and furthermore subject to some additional constraints, e.g., on the amplitude of the motion. The thrust is obtained by summing the integrated x-component of the pressure jump across the profile and the suction force at the leading edge. The suction force always acts as a positive thrust. The relative contribution of the suction force to the total thrust will be constrained. The reason is that in certain cases it can be shown that without this constraint optimum motions do not exist. The constraint on the suction is also useful from the mechanical point of view, because large suction forces cause the separation of flow from the profile.

Furthermore, from the engineering point of view it is desirable to constrain the amplitude of a point of the profile. Hence the optimization problem that we shall consider is for given $\overline{T} > 0$, $r > 0$ and $C_\infty > 0$,

$$\begin{aligned} &\underset{h \in \mathcal{H}}{\text{minimize}}\ E(h), \quad \text{subject to } T(h) = \overline{T},\ T^s(h) \le r\overline{T}, \\ &\max_t |h(x_p, t)| \le C_\infty, \end{aligned} \tag{2.3}$$

where $\mathcal{H}$ is the function space in which the optimum motion is sought, $T^s(h)$ is the mean suction force, and x_p is the x-coordinate of the point whose amplitude is constrained.

We assume that the motions have small amplitudes so that a linearized theory can be applied. In this theory all flow quantities can be explicitly expressed in terms of a and α, more precisely in terms of their Fourier coefficients. By writing

$$\begin{aligned} a(t) &= \sum_{n=-\infty}^{\infty} \hat{a}(n) \exp(2\pi i n t/\tau_0), \\ \alpha(t) &= \sum_{n=-\infty}^{\infty} \hat{\alpha}(n) \exp(2\pi i n t/\tau_0), \end{aligned} \tag{2.4}$$

we find for example

$$E(h) = \sum_{n=-\infty}^{\infty} (\hat{a}(n), \hat{\alpha}(n))\, \mathcal{E}(n\sigma_0) \begin{pmatrix} \hat{a}(n) \\ \hat{\alpha}(n) \end{pmatrix}^*, \tag{2.5}$$

where * denotes complex conjugation, $\sigma_0 = 2\pi\ell/(\tau_0 U)$ and $\mathcal{E}(\sigma)$ is for all $\sigma \neq 0$ a nonnegative selfadjoint $(2,2)-$matrix, and therefore E is a convex quadratic functional. Analogous expressions hold for $T(h)$ and $T^s(h)$ with selfadjoint matrices $\mathcal{T}(\sigma)$ and $\mathcal{T}^s(\sigma)$ instead of $\mathcal{E}(\sigma)$. $\mathcal{T}^s(\sigma)$ has one positive and one vanishing eigenvalue, whereas $\mathcal{T}(\sigma)$ has one positive and one negative eigenvalue, in agreement with the fact that the suction is always nonnegative, whereas the total thrust can be negative as well as positive. Hence T^s is convex, whereas T is not.

In order to prevent unessential constraints on the smoothness of the motions, one should choose for $\mathcal{H}$ the largest function space for which all functionals occurring in the optimization problem are well-defined and norm-continuous. Because the nonzero eigenvalues of the matrices $\mathcal{E}(\sigma)$ and $\mathcal{T}^s(\sigma)$ are $\sim \sigma^2$ for $\sigma \to \pm\infty$, this means that we require a, and α to be in the Sobolev space $H^1_{\tau_0}$ defined by

$$\begin{aligned} H^1_{\tau_0} = \{ f \in L^2_{loc}(\mathbf{R});\ f' \in L^2_{loc}(\mathbf{R}), \quad \text{and} \\ f(t+\tau_0) = f(t) \text{ for all } t \}. \end{aligned} \tag{2.6}$$

With the standard scalar product $H^1_{\tau_0}$ is a Hilbert space. It is well known that the functional $h \mapsto \max|h(x_p,t)|$ is continuous with respect to the weak topology of this space.

In studying the existence of optimum motions it is natural to attempt to apply the general theorem which says that a lower semi-continuous (l.s.c.) functional attains its infimum on a compact set. Now the constraint set in

(2.3) is bounded and norm-closed, but not norm-compact. One would like to choose a smaller topology on the space $H^1_{\tau_0}$, such that the set is compact and such that E is l.s.c. Because E is convex and norm-continuous, it is l.s.c. with respect to the weak topology on $H^1_{\tau_0}$, and this weak topology is in practice the smallest topology for which E has this property. Nevertheless, the set in (2.3) is not compact in this topology either. As is often the case in infinite-dimensional optimization problems of hydrodynamic propulsion, the trouble is caused by the equality constraint on the mean thrust.

In spite of the fact that the set in (2.3) is not compact in the weak topology, it is possible to prove the existence of an optimum motion. The idea of the proof is based on the important observation that the difference $G(h)$ between the useful work and the lost energy:

$$G(h) = UT(h) - E(h) \tag{2.7}$$

can be shown to be weakly continuous. Problem (2.3) is equivalent to

$$\begin{aligned} &\text{maximize } G(h), \quad \text{subject to } T(h) = \overline{T}, \quad T^s(h) \leq r\overline{T}, \\ &a, \alpha \in H^1_{\tau_0} \\ &\max_t \; |h(x_p, t)| \leq C_\infty. \end{aligned} \tag{2.8}$$

In addition to (2.8) we introduce the optimization problem obtained by replacing the equality constraint on the generated thrust by an inequality:

$$\begin{aligned} &\text{maximize } G(h), \quad \text{subject to } T(h) \leq \overline{T}, \quad T^s(h) \leq r\overline{T}, \\ &a, \alpha \in H^1_{\tau_0} \\ &\max_t \; |h(x_p, t)| \leq C_\infty. \end{aligned} \tag{2.9}$$

This problem is not necessarily equivalent to (2.8), because it could have a solution with $T(h) < \overline{T}$. It follows from $UT(h) = G(h) + E(h)$ and from the mentioned properties of G and E that T is l.s.c. with respect to the weak topology. Because T^s is convex and norm-continuous, it has this property also. Therefore, the set in (2.9) is weakly closed and since it can be shown to be bounded also, it is weakly compact. We conclude therefore that problem (2.9) has at least one solution. Furthermore, one can prove that at least one solution satisfies $T(h) = \overline{T}$ and therefore is also a solution of problem (2.8). Because problems (2.3) and (2.8) are equivalent, we conclude that optimization problem (2.3) has indeed at least one solution.

When the required mean thrust is smaller than a treshold value $\overline{T}_1$, the solution is a pure harmonic with lowest frequency $1/\tau_0$. When the required mean thrust is larger than $\overline{T}_1$, the constraint on the amplitude of the motion becomes active and both the heaving and the pitching components consist of infinitely many nonvanishing harmonics. The amplitude constraint is active for two intervals of time per period during which the point x_p is at rest at

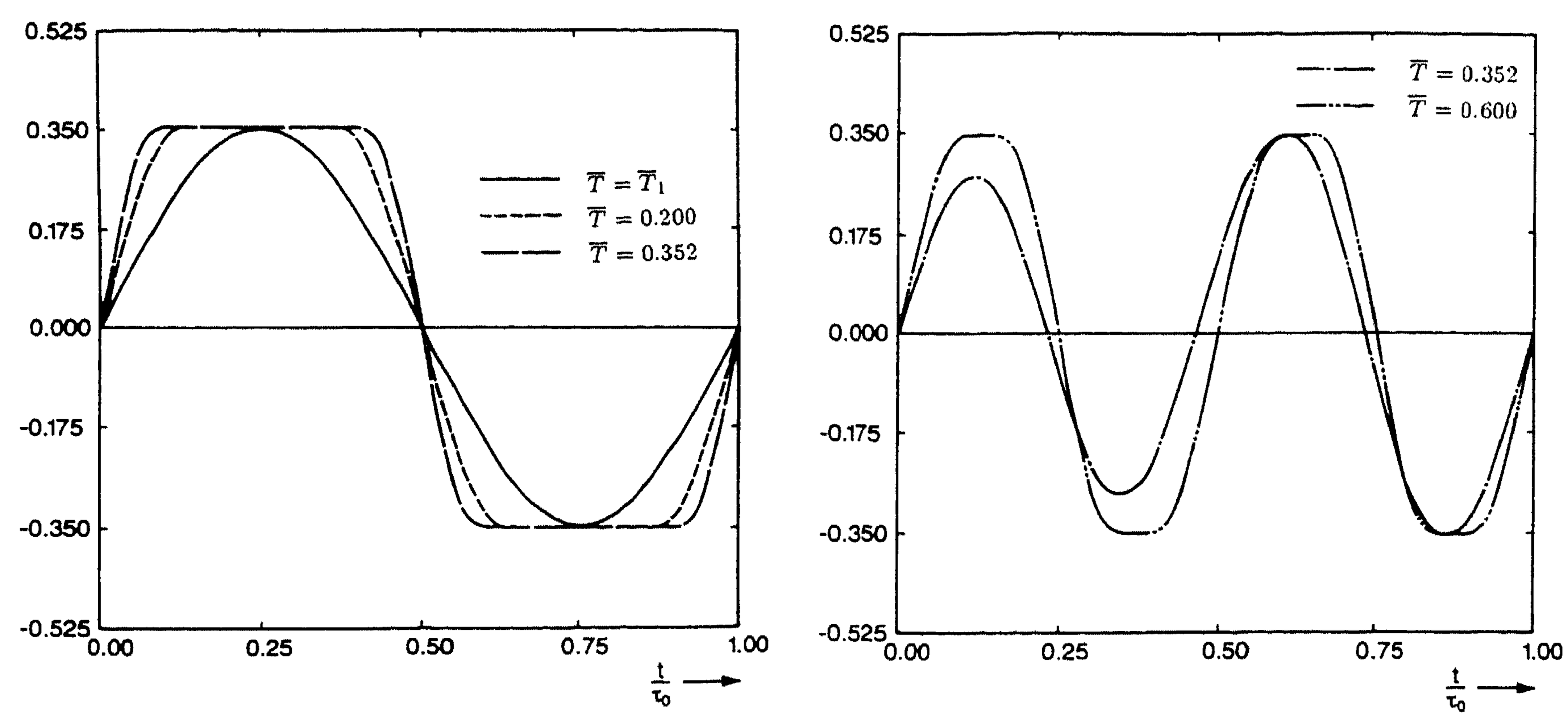

Figure 3. The motion of the quarter-chord point as function of time for optimum motions corresponding to $\sigma_0 = \pi/3$, $x_p = 0.5l$, $r = 0.4$, $C_\infty = 0.35l$ and for four values of the required mean thrust: $\overline{T} = \overline{T}_1 = 0.066$, $\overline{T} = 0.200$, $\overline{T} = 0.352$ and $\overline{T} = 0.600$. All optimum motions shown in the left figure have dominant lowest harmonic, whereas those shown at the right have dominant second harmonic.

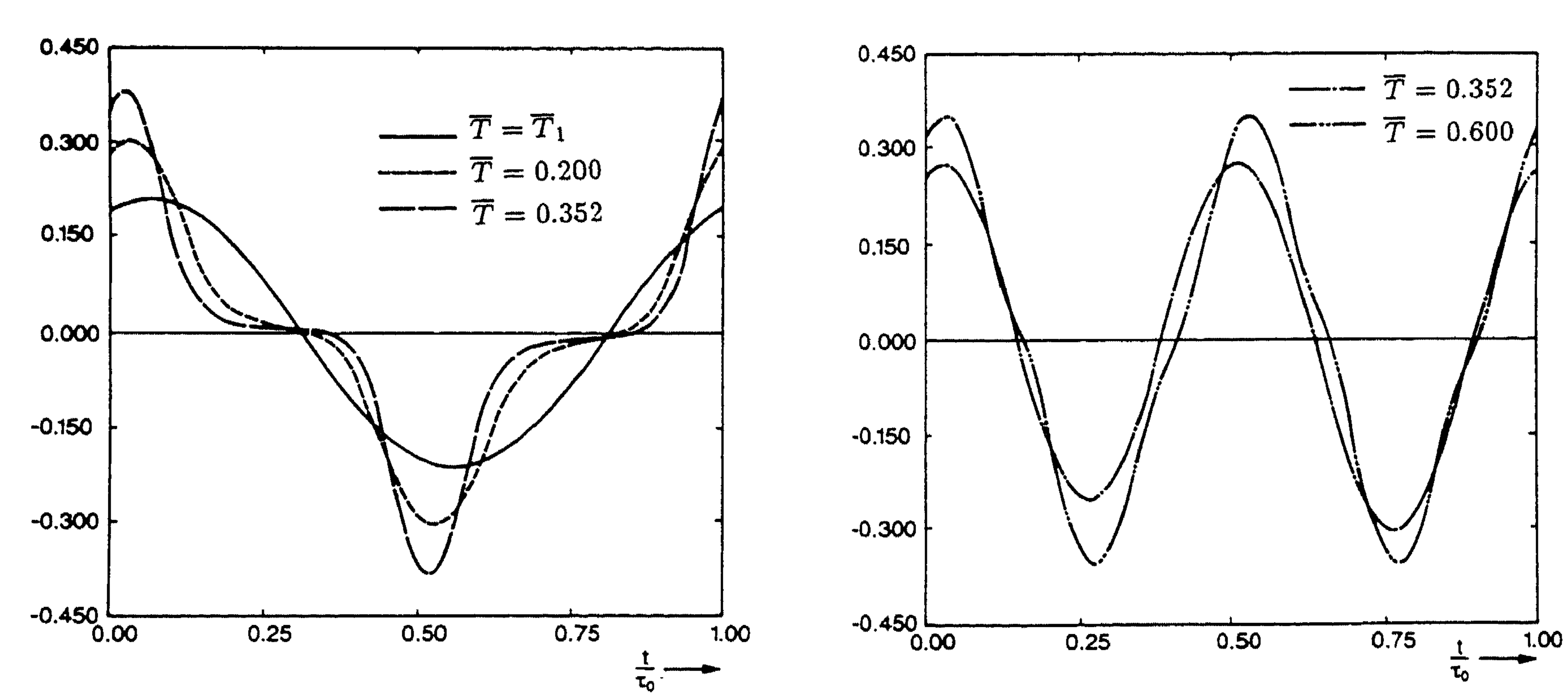

Figure 4. The pitching $\alpha(t)$ of the optimum motions of figure 3.

the maximum stroke while the profile slowly pitches around it. For $r = 0.4$, $x_p = 0.5\ell$ (quarter-chord point), $C_\infty = 0.3525\ell$ and for several values of the thrust, $h(x_p, t)$ and the pitching $\alpha(t)$ of the optimum motion are shown in figures 3 and 4. For $\overline{T}_1 \leq \overline{T} < 0.3525$ the lowest harmonic is dominant. For $\overline{T} = 0.3525$, a second optimum exists having dominant second harmonic, and for larger required thrust the second harmonic remains dominant until the third takes over at a certain higher treshold value. All computed optimum motions generate the maximum allowed mean suction force $r\overline{T}$, even when r is much larger than 1.

Several properties of the optimum motions can be derived from a Lagrange multiplier rule that is obtained by the application of a Kuhn-Tucker type of theorem (which provides a method to solve certain minimization problems with inequality constraints). However, the classical Kuhn-Tucker Theorem cannot be used because the functional $h \mapsto \max|h(x_p, t)|$ is not Gateaux differentiable. When this type of constraint occurs, the theory of generalized differentials has to be applied.

3. Optimum large amplitude sculling propulsion with an inequality constraint on the side force

Next, we consider the large amplitude motion of a lifting line [2], which can represent a one-wing sculling propeller, mounted vertically at the stern of a ship (see figure 5). The sculling wing W moves sideways back and forth, while its angle is adjusted such that a thrust is created. From the hydrodynamical point of view, a large lateral amplitude is profitable, since this can result in a high efficiency.

The lifting line or concentrated bound vortex $\Gamma(t)$, which represents this sculling wing, moves through the water along a line G (see figure 6). The strength of the vortex varies with time, corresponding with the blade angle variation of the wing. Since the bound vorticity $\Gamma(t)$ varies with time, free vorticity is shed into the water. In other words, the fluid behind the lifting line is put into motion and its kinetic energy increases with time. It is clear that the lost kinetic energy should be kept as small as possible.

By the motion of the lifting line a 'lift' force is evoked, acting perpendicularly to the local direction of motion, hence normal to G. By Joukowski's law, the magnitude of this force is proportional to the product of the vortex strength and the velocity of the lifting line. The lift force can be decomposed into two components: a thrust component T in the direction of motion of the ship and a side force component S perpendicular to it. The mean value of the thrust should be equal to the ship's hull drag at a desired speed. The most efficient propulsor is the one that produces the least kinetic energy under the constraint of a prescribed mean thrust.

The fluctuating side force is an evident drawback of a one-wing sculling propeller. It can have a disturbing influence on the course of the ship.

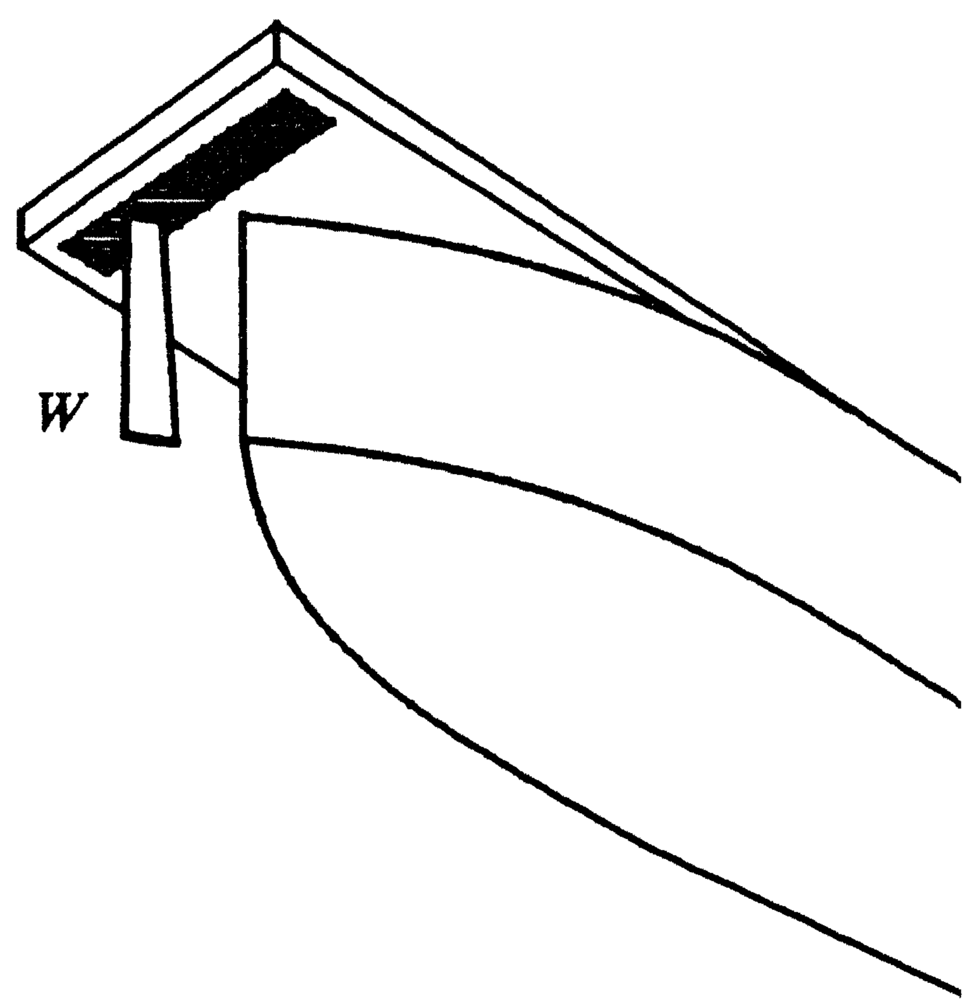

Figure 5. Stern of a ship, equipped with a one-wing sculling propeller W.

Therefore, it is studied what the effect is on the optimum motion if a constraint is put on this lateral force. To be more precise, a maximum is put on its absolute value. The optimization problem we consider is to find an optimum time-dependent bound vorticity of the lifting line for which the kinetic energy generated per period of time is minimal, under the constraints of a thrust with a prescribed mean value and a side force with a maximum value. Contrary to the previous section, the path G is chosen in advance.

The problem that we are confronted with consists of two parts. First, it has to be proved that an optimum bound vorticity *exists.* Second, the optimum bound vorticity has to be *constructed.* In the following, an outline is given of the procedure that is followed to solve these problems and which role is played by functional analysis. Since the mathematical implications are rather complex, a two-dimensional model is used. This means that the lifting line is assumed to be infinitely long, having a constant strength in spanwise direction. Furthermore, a linearized theory of an inviscid fluid is adopted, implying that the shed free vorticity keeps its strength and remains on the place where it is formed, that is on G.

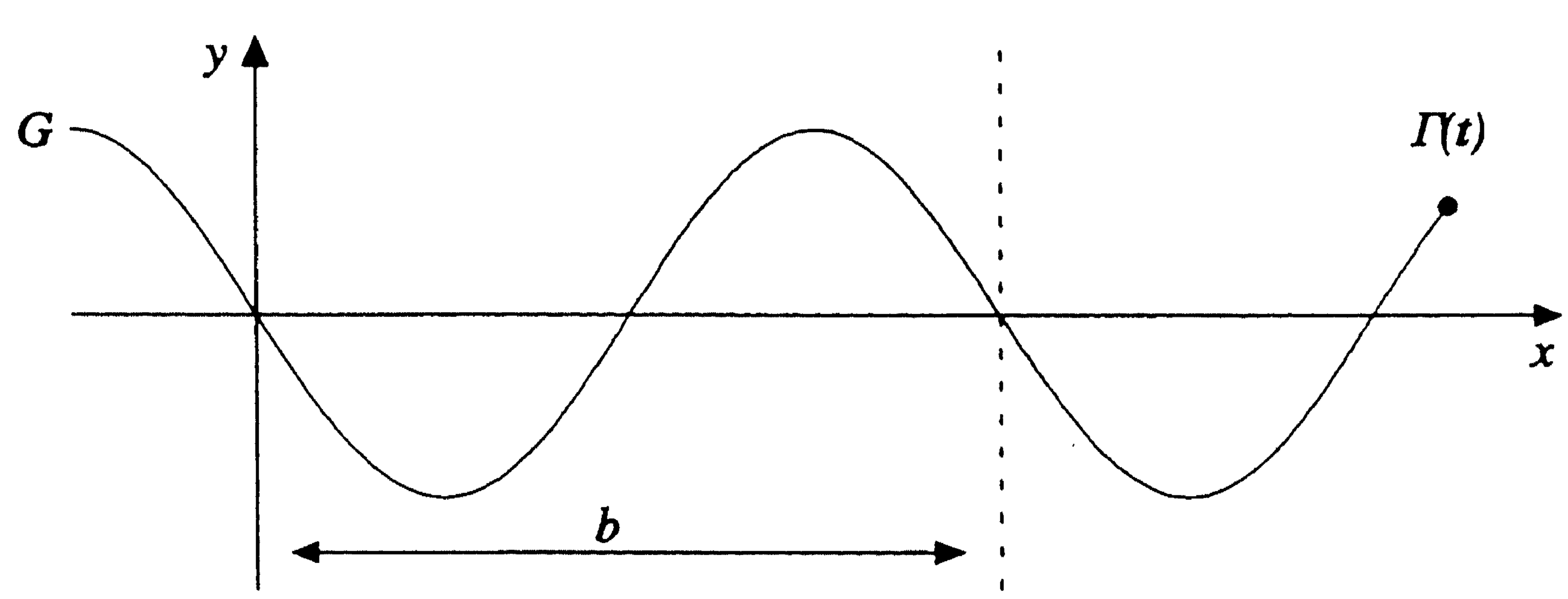

Figure 6. The motion of the lifting line $G(t)$.

The first step, before we can prove the existence of an optimum motion, is to state the optimization problem unambiguously in a proper mathematical sense. For that purpose, let us consider a Cartesian reference frame x, y, in which the positive x-direction is defined by the direction of motion of the ship (which is assumed to move in a straight line).

Now let us say that we find ourselves at a certain position x, y. The ship passed by a very long time ago and disappeared behind the horizon ($x = \infty$), so we do not experience the unsteady motion of the lifting line. On the other hand, the ship started its motion somewhere behind the opposite horizon ($x = -\infty$), so starting effects can be neglected as well. What remains is a steady flow induced by the free vorticity on G. During one time period of the wing motion, an amount of kinetic energy is added to the fluid which is equal to the kinetic energy in the strip:

$$\Omega = \{(x, y) \in \mathbb{R}^2 \,; 0 \leq x \leq b\}, \tag{3.1}$$

where b is the distance covered by the ship during one period.

Returning to our steady state model, the only vorticity that is present in the fluid is on G. So, outside G a velocity potential ϕ exists. The kinetic energy, produced per time period can then be expressed as:

$$E(\phi) \int\int_{\Omega} \| \nabla \phi \|^2 dx dy. \tag{3.2}$$

The potential ϕ is not continuous over G. The free vortices on G induce a jump $[\phi]$, which is such that the derivative of $[\phi]$ along G is equal to the strength of the free vorticity. We can choose the potential ϕ such that the jump $[\phi]$ at a certain location on G is equal to the vortex strength $\Gamma(t)$ of the lifting line at the time it passed by.

Since the location of G is prescribed, it is allowed to treat $[\phi]$ as a function of x only. The constraint of the prescribed mean thrust can be expressed as a weighted integral of $[\phi]$, symbolically written as:

$$T([\phi]) = \bar{T}. \tag{3.3}$$

The inequality constraint on the side force can be expressed, directly in terms of $[\phi]$, as:

$$-\bar{S} \leq [\phi](x) \leq \bar{S}, \text{ for every } x. \tag{3.4}$$

Summarized, our aim is to find a potential ϕ, satisfying (3.3) and (3.4), for which the lost energy $E(\phi)$, (3.2), is as small as possible.

To define the optimization problem well in the mathematical sense, we have to identify a convenient function space for ϕ, in which the optimization can be carried out. This space has to be such that (3.2), (3.3) and (3.4) are well-defined. An appropriate candidate for this is the Sobolev space $H^1(\Omega)$, consisting of square-summable functions on Ω, of which the derivatives are square-summable too. In this space, (3.2) is automatically well-defined.

At the upper and lower side of G, boundary values of $\phi \in H^1(\Omega)$ exist in the sense of the so-called 'trace'. These boundary values are elements of the Sobolev space $H^{1/2}(0,b)$, which is defined using derivatives of non-integer order. Since $H^{1/2}(0,b)$ is contained in the space of square-summable functions $L_2(0,b) = H^0(0,b)$, it is clear that (3.3) is also well-defined. The side force constraint (3.4) has to be slightly weakened as:

$$-\bar{S} \leq [\phi](x) \leq \bar{S}, \text{ for } \textit{almost} \text{ every } x. \tag{3.5}$$

This is because Sobolev spaces, in fact, consist of *equivalence classes* of functions.

So, we can formulate the problem as the minimization of $E(\phi)$, (3.2), on the set

$$P = \{\phi \in H^1(\Omega);\ \phi \text{ satisfies (3.2) and (3.4)}\}. \tag{3.6}$$

To prove the existence of a solution we use the well-known fact that a closed, convex subset of a Hilbert space possesses a unique element which minimizes the norm. Indeed, P is a closed, convex subset of the Hilbert space $H^1(\Omega)$. Here we benefit from the fact that the path G is fixed, in contrast with the previous section, where the counterpart of P is *not* convex.

The energy $E(\phi)$ is in general not equivalent with the H^1-norm. However, if we equip $H^1(\Omega)$ with some evident symmetry and periodicity properties, then $E(\phi)$ can be proved to be equivalent with the usual norm on $H^1(\Omega)$. By this strategy, the existence of an optimum motion of the lifting line is proved. Moreover, it follows that there is *only one* optimum motion.

It is noted that, up to now, we have completely ignored the incompressibility of the fluid. It turns out, however, that this omission is not essential. By disturbing the optimum potential ϕ_0 with test functions which are continuous over G, it is seen that $\Delta\phi_0 = 0$ (Δ is the Laplace operator), which means that the velocity field of the optimum potential is free of divergence. Moreover, it follows that the normal velocity is continuous across G, as it should be. So, although we admitted divergence, the solution of the optimization problem is *free* of divergence.

One of the constraints, namely (3.5), is defined on an infinite set. Consequently, one of the Lagrange multipliers is not a scalar, but a function. Then, it is not evident that (a generalized version of) the Kuhn-Tucker theorem is applicable. The classical generalized Kuhn-Tucker theorem requires the set P, in which the optimization is carried out (cf. (3.6)), to be a so-called 'positive cone' (a semi-infinite set). Furthermore, for P the 'regularity condition' should hold, which means that this cone must have a non-empty interior. It is indeed possible to reformulate the optimization problem, such that P is a positive cone. However, the set of positive functions of $H^{1/2}(\mathbb{R})$, the space in which (3.5) is defined, has *no* interior points.

To overcome this difficulty, an other version of the Kuhn-Tucker theorem, without regularity condition, would be useful. However, in the open literature some variants were found. However, in these theorems the regularity condition was replaced by other, complicated conditions, of which the validity could not be proved in our situation. Therefore, a new Kuhn-Tucker theorem was developed [3] with less complicated requirements, which appeared to hold in our optimization problem.

Herewith, the optimization problem is solved. In other words, in our simple, two-dimensional model, a unique optimum motion *exists* of a sculling wing (represented by a lifting line) with prescribed mean thrust and bounded side force. Moreover, we are able to *construct* this optimum motion.

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